Simultaneous Innovation and the Cyclicality of R&D

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September 5, 2019

Abstract

There is ample evidence that R&D investment is mildly pro-cyclical. Whereas the existing literature can explain the positive correlation between investment in R&D and output, the moderate strength of the relationship remains under-explored. This paper develops a stochastic expanding-variety endogenous growth model that accounts for the observed mild pro-cyclicality of R&D. In the model, several firms may simultaneously make the same innovation. Innovations made by many firms simultaneously are of higher quality, on average, and contribute relatively more to the expansion of the knowledge stock in the economy. This delivers an endogenous mechanism that breaks the otherwise perfect correlation between R&D and output. A calibration of our model closely matches the cyclical properties of R&D.

Keywords: Simultaneous Innovation, Research and Development, Medium-Term Cycles, Macroeconomic Fluctuations, Endogenous Cycles.

JEL Codes: O30, O40, E32.

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1 Introduction

It is a well-established empirical regularity that R&D investment is mildly pro-cyclical.[1] Depending on the data source, the correlation between the cyclical components of R&D and output is between 0.3 and 0.5.[2] Whereas the existing literature has proposed several mechanisms that can explain the sign of this relationship, its magnitude remains under-explored.[3] In particular, existing theoretical models predict near-perfect correlation between output and R&D, which is at odds with the data.[4]

This paper complements the existing literature by developing an expanding-variety endogenous growth model that can resolve the discrepancy. In our model, there is the possibility that several firms make the same innovation simultaneously. Innovations simultaneously made by many firms contribute more, on average, to the expansion of the knowledge stock in the economy, i.e. they are of higher quality. This mechanism delivers endogenous oscillations in the economy and, as a consequence, mildly pro-cyclical R&D. A calibrated version of our model closely matches the pro-cyclicality and volatility of R&D observed in the data.

The source of technological progress in our model is the invention and adoption of new intermediate varieties. As in Gabrovski (2018) and Kultti *et al.* (2007), the innovation process makes the distinction between potential innovations (that we refer to as ideas and research avenues interchangeably) and actual innovations (new varieties). Upon entry into the R&D sector each firm is randomly matched with a particular idea from a pool of feasible research avenues. In particular, the number of firms matched with a given idea is a Poisson random variable. This matching technology generates the possibility that some ideas are simultaneously innovated by many firms while others are not innovated at all.[5] This feature allows our model to account for the commonly observed in practice phenomenon of simultaneous

[1]See, for example, Griliches (1990), Fatas (2000), Wälde and Woitek (2004), Comin and Gertler (2006), Barlevy (2007), Ouyang (2011), Fabrizio and Tsolmon (2014), and Sedgley *et al.* (2018).

[2]For further details see, for example, Comin and Gertler (2006), Francois and Lloyd-Ellis (2009), and section three of this paper.

[3]See, for example, Fatas (2000), Comin and Gertler (2006), Barlevy (2007), and Francois and Lloyd-Ellis (2009).

[4]This is the case in both papers that develop expanding-variety growth models (see, for example, Comin and Gertler (2006)) and papers that develop Schumpeterian growth models (see, for example, Francois and Lloyd-Ellis (2009)).

[5]For previous growth models that feature simultaneous innovation in the same sector or of the same idea see, for example, Corriveau (1998) and Gabrovski (2018).

innovation.[6] When a firm is matched with an idea and successfully innovates that idea, it applies for a patent over the corresponding variety. If more than one firm apply for the same patent, they each have an equal chance of receiving it. We follow Romer (1990) and Kortum (1997), among others, and assume that knowledge is cumulative — inventing a new variety allows firms to "stand on the shoulders of giants" and gain technological access to a number of new ideas.

Each variety is equally productive, but innovations differ in the number of new research avenues they generate, i.e. their "quality". In particular, innovations made simultaneously by many firms lead to more research avenues, on average. This captures the intuition that i) when firms invest more in a given idea, it is more likely to be of higher quality; ii) when more firms work simultaneously on the same idea, it is more likely that at least one of them will develop a high quality invention.

The main contribution of our paper is the model's ability to reproduce the mild pro-cyclicality of R&D observed in the data. In our model the correlation between output and R&D is 0.44 and R&D is 1.89 times as volatile as output. In the data, the correlation is $0.3 - 0.5$ and the relative volatility of R&D is $1.79 - 1.9$ times that of output.[7] In our model, innovations made simultaneously by many firm are, on average, of higher quality. This generates a mechanism which leads to endogenous oscillations. As a result, R&D investment is mildly pro-cyclical. In particular, following a positive technology shock both R&D and output converge to their new, higher balanced growth paths (BGP henceforth). Thus, their cyclical components are positively correlated. However, during this transition both series oscillate around their convergent paths in such a way that whenever output overshoots its convergent path, R&D investment undershoots it and vice versa. This reduces the strength of the relationship and leads to mildly pro-cyclical R&D.

To see the intuition behind the endogenous oscillations, suppose that at a given period, τ, the economy features relatively more varieties and relatively less available research av-

[6] For example, on February 14, 1876 Alexander Bell and Elisha Gray applied for a patent over the telephone within hours of each other. This same phenomenon is observed with virtually every major innovation from history, such as the cotton gin, the steam engine, the laser, and the computer (see, for example, Lemley (2011)). Furthermore, instances of simultaneous innovation have also been documented in many cases for non-major innovations as well (see, for example, Cohen and Ishii (2005) and Gabrovski (2018)).

[7] See Comin and Gertler (2006), Francois and Lloyd-Ellis (2009) and section three of this paper.

enues. This scarcity of ideas implies that there will be few innovations made and, as a result, relatively less varieties next period. This relatively low number of varieties at $\tau + 1$ implies lower competition between firms and increases the expected profitability of innovation. Hence, firms have more incentives to enter the R&D sector. This in turn leads to higher congestion in the "market" for ideas (i.e. relatively higher ratio of firms to ideas) and to a higher average number of firms that simultaneously innovate the same idea. Because of this, the average quality of innovations is higher, which leads to more feasible research avenues at period $\tau + 1$. Thus, at period $\tau + 1$ the economy features relatively more ideas and less varieties. As a result, at that period, there is relatively more innovation which leads to more varieties at $\tau + 2$. This then leads to lower expected profits and lower incentives for firms to enter the R&D sector, which ultimately leads to lower mass of ideas at $\tau + 2$. Furthermore, in periods when there are more varieties output is relatively high, whereas R&D investment is relatively low because research avenues are scarce. Conversely, when ideas are relatively abundant R&D investment is high and output is low because such periods feature a relatively lower mass of varieties.

We proceed by introducing the environment and characterizing the equilibrium. Next, we simulate the model and examine its impulse response functions and the oscillations therein. Lastly, we show our model can match key moments in the data and this ability is driven by the presence of endogenous oscillations.

2 The Economy

There are three types of agents in the economy — a final good producer, a unit measure of consumers (households), and a continuum of R&D firms. Time is discrete and infinite. The final good firm employs capital, labor, and intermediate varieties, which it uses to produce a single final good. Consumers supply labor, own the capital stock and the R&D firms, and consume the final good. R&D firms employ labor and engage in innovative activities. Firms that successfully innovate and patent a variety produce that variety.

2.1 Final Good Sector

The final good, Y_t, is produced by a single price taker. The price of the final good is normalized to unity. We follow Comin and Gertler (2006) and endow the firm with the following technology:

$$Y_t = A_t(K_t^{\alpha} L_{P_t}^{1-\alpha})^{1-\sigma}\Big(\int_0^{N_t} X_t^{\lambda}(n)dn\Big)^{\frac{\sigma}{\lambda}}, \quad \alpha, \sigma, \lambda \in (0,1) \tag{1}$$

The firm rents capital, K_t, from households at the rate $r_t + \delta^K$, where δ^K is the depreciation rate of capital and r_t is the households' rate of return. The firm faces a competitive market for labor in production, L_{P_t}, which is hired at the wage w_t. $X_t(n)$ is the amount of a particular variety n employed in production and N_t is the mass of intermediate varieties. The final good firm faces a monopolistically competitive market for these varieties, where a unit of each variety n is bought at the price $P_t(n)$.

We follow the RBC literature and assume the only source of aggregate uncertainty in the model is a productivity shock. In particular, the productivity parameter, A_t, follows an AR(1) process in logs:

$$A_{t+1} = A_t^{\rho} u_{t+1} \tag{2}$$

where $\rho \in (0,1)$ is a persistence parameter and u_{t+1} is a unit mean shock with variance σ_u.

The usual profit maximization of the final good firm implies the following demand functions for labor in production, capital, and intermediate varieties:

$$w_t = (1-\alpha)(1-\sigma)\frac{Y_t}{L_{P_t}} \tag{3}$$

$$r_t = \alpha(1-\sigma)\frac{Y_t}{K_t} - \delta^K \tag{4}$$

$$P_t(n) = \sigma X_t^{\lambda-1}\frac{Y_t}{\int_0^{N_t} X_t^{\lambda}(n)dn} \tag{5}$$

2.2 R&D Sector

The novel features of our model are contained within the R&D sector. Figure 1 illustrates the timing of the R&D process and the process by which ideas and varieties accumulate. The

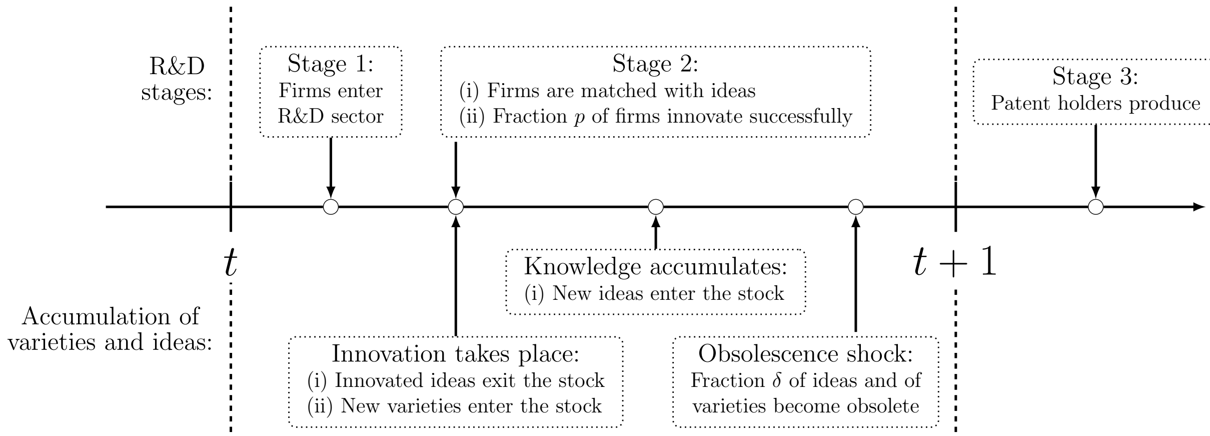


Figure 1: Timing of R&D and Accumulation of Ideas and Varieties

research and development process consists of three stages and makes the distinction between potential innovations (that we refer to as ideas and research avenues interchangeably) and actual innovations (new varieties). At stage one firms enter the R&D sector. At stage two, they are matched with ideas — each firm is matched with a single idea, but the number of firms matched with a particular idea is a random variable. As a consequence, several firms may innovate the same idea simultaneously. The more firms there are relative to ideas, the more likely it is simultaneous innovation will take place. If innovated, an idea transforms into a new variety. Firms that innovate successfully and secure a patent produce at stage three.

The R&D process is closely linked to the growth of the stock of ideas and varieties that takes place in the economy. Invented ideas transform into new varieties and so innovation expands the stock of the latter. Due to knowledge accumulation, innovation increases the stock of research avenues as well. Inventing a new variety allows firms to "stand on the shoulders of giants" and gain technological access to new research avenues. The number of new ideas that come about from an innovation represents its quality. Innovations made by more firms simultaneously expand the stock of ideas by relatively more and so we say they are of higher quality. The number of firms that innovate the same idea simultaneously is a random variable with an endogenously determined distribution. Thus, the distribution of innovation quality is endogenous as well.

R&D process. At stage one, firms enter the R&D sector at a cost $\eta/(N_t A_t)$ units of labor. The entry cost depends on both the mass of intermediate varieties in the economy, N_t, and aggregate productivity, A_t. As in Romer (1990), among others, the cost of innovation decreases as the mass of varieties expands, which allows the model to exhibit positive long-run growth. In the spirit of the RBC literature, our model features an economy-wide productivity shock. To this end, we follow Bilbiie *et al.* (2012) and assume that the entry cost is decreasing in A_t. Let μ_t be the mass of R&D entrants and L_{R_t} be the total amount of labor employed in R&D. Then, the economy-wide research production function is

$$\mu_t = A_t N_t L_{R_t} \eta^{-1} \tag{6}$$

At stage two, each firm is matched with a particular idea from a finite mass ν_t of feasible research avenues. Ideas are ex-ante identical. Every firm is matched with a single idea but the number of firms matched with a given idea is a random variable that follows a Poisson distribution with mean equal to the tightness in the market for ideas, $\theta_t = \mu_t/\nu_t$. Once matched, each firm innovates successfully with probability p. With probability $1-p$ the firm fails and so it exits the innovation sector. Thus, the chance an idea is invented by m firms is $\sum_{n=m}^{\infty} Pr(n$ firms are matched with the idea$) \times Pr(m$ firms are successful in innovating $\mid n$ firms are matched$) = \sum_{n=m}^{\infty} \frac{\theta_t^n e^{-\theta_t}}{n!} \binom{n}{m} p^m (1-p)^{n-m} = \frac{(p\theta_t)^m e^{-p\theta_t}}{m!}$. As a consequence, the number of firms that successfully innovate a particular idea is a Poisson random variable with mean $p\theta_t$. Due to the uncertainty in the matching process, some research avenues are innovated by many firms simultaneously while others are not innovated at all, but on average $p\theta_t$ firms innovate the same idea. The distribution of firms that innovate the same idea simultaneously is endogenous — if more firms enter the R&D sector, the tightness, θ_t, increases which raises the likelihood of simultaneous innovation.

If an idea is invented, it transforms into exactly one new variety. Firms that successfully innovate, apply for a patent over the corresponding variety. In the event that several firms innovate the same idea simultaneously, each receives the patent with equal probability. Then, the chance that a patent application is granted is $\sum_{m=0}^{\infty} Pr($exactly m rivals successfully innovate the same idea$)/(m+1) = \sum_{m=0}^{\infty} e^{-p\theta_t}(p\theta_t)^m/(m+1)! = (1-e^{-p\theta_t})/(p\theta_t)$. Higher tightness increases the chance more firms innovate the same idea simultaneously, so each has a lower probability of securing the patent.

At stage three, firms that have secured a patent over a variety produce it in a monopolistically competitive market. Implementation of the new variety takes one period, so firms that successfully innovate at period t begin production at period $t+1$. Producing one unit of a variety requires one unit of the final good, so per-period profits are $\pi_t(n) = (P_t(n) - 1)X_t(n)$. Patents grant perpetual monopoly, but varieties become obsolete with probability δ.[8] Thus, the value of holding a monopoly over a variety n at time t, $V_t(n)$, is

$$V_t(n) = E_t \sum_{i=t+1}^{\infty} (1-\delta)^{i-t} d_{it} \pi_i(n) \tag{7}$$

where d_{it} is the stochastic discount factor.

Knowledge accumulation. A firm that innovates successfully submits a patent application. The application contains technological know-how — a method by which one can produce the corresponding variety. After it is submitted, rival firms can observe and read the patent application. Thus they acquire the technological know-how described in the application and, as a result, gain access to new research avenues.[9] Hence, whenever an idea is innovated, new ones are added to the stock. The number of ideas added to the pool due to an innovation represents its quality: high quality innovations grant access to many research avenues, low quality ones to a few.

When several firms innovate simultaneously, they all submit a patent application for the same innovation. Although all of these applications contain information on how to produce the new variety, the methods described may differ. That is, each patent application contains different know-how. In addition, some applications may describe a method that is widely applicable and grants researchers access to many new ideas, whereas some may describe a method that is of limited interest to scientists and grants access to only a few avenues for research. As a result, the more firms submit a patent application for the same innovation the higher the quality of that innovation on average. This process of knowledge accumulation is

[8] Given the timing in our model, it is possible that a variety innovated at time t becomes obsolete even before production begins at $t+1$.

[9] This is in the spirit of Kortum (1997) and Romer (1990), among others, who also assume that knowledge is cumulative.

consistent with the intuition that i) the higher is the R&D investment on an idea, the higher its expected quality; ii) whenever more firms work on the same idea, it is more likely at least one of them develops a high quality invention.[10]

The number of firms that innovate the same idea simultaneously is a random variable with an endogenous distribution. As a result, so is the quality of an innovation. Intuitively, if more firms enter the R&D sector, the market becomes more congested and the tightness increases. Higher θ_t raises the likelihood more firms will make the same innovation simultaneously, which in turn increases the probability that innovation will be of higher quality. For the purposes of our analysis we do not need to keep track of the whole distribution of innovation quality but rather just of its mean, denoted by $M(\theta_t)$. To derive our analytical results, it is sufficient to impose the following two restrictions: i) $M'(\theta_t) > 0$; ii) $M(\theta_t) > 1$ for all positive θ_t. The first assumption captures the intuition that if more firms innovate the same idea on average, then the average innovation quality is higher. The second assumption is a necessary condition for positive long-term growth in our economy. To obtain our numerical results, we study a loglinearized version of the model. To that end we only need to know the value of $M(\theta_t)$ and its elasticity along the deterministic BGP, both of which we calibrate directly. Thus, we keep it general and do not endow $M(\theta_t)$ with a specific functional form.[11]

Laws of motion for ideas and varieties. First, we turn to the law of motion for ideas. In a given period, three events, as depicted in Figure 1, affect the accumulation of research avenues: i) innovation; ii) knowledge accumulation; iii) an obsolescence shock. At the beginning of period t, the initial stock of ideas is ν_t. Given the matching technology, the probability a research avenue is invented equals $1 - Pr(0 \text{ firms innovate the idea}) = 1 - e^{-p\theta_t}$. Innovated ideas are transformed into new varieties and as a consequence exit the stock of research avenues. Thus, after innovation takes place the remaining mass of

[10]If an idea is matched with m firms, the R&D investment on that idea is $m\eta w_t/(N_t A_t)$ and it is simultaneously invented by pm firms on average.

[11]Conditional on being innovated, an idea is invented by m firms simultaneously with probability $e^{-p\theta_t}(p\theta_t)^m/(m!(1-e^{-p\theta_t}))$. Let $\iota(m)$ be the average number of new research avenues generated if that event occurs. Then, the expected number of new ideas that come about from the innovation is $M(\theta_t) \equiv \sum_{m=1}^{\infty} \iota(m) Pr(\text{ idea is invented by } m \text{ firms } | m \geq 1) = \sum_{m=1}^{\infty} \iota(m) e^{-p\theta_t}(p\theta_t)^m/[m!(1-e^{-p\theta_t})]$. Thus, the functional form of $M(\theta_t)$ depends on the functional form of $\iota(m)$. For example, if $\iota(m) = \kappa m$, then $M(\theta_t) = \kappa p\theta_t/(1-e^{-p\theta_t})$ and if $\iota(m) = \kappa^m$, then $M(\theta_t) = (e^{(\kappa-1)p\theta_t} - e^{-p\theta_t})/(1-e^{-p\theta_t})$. In our analysis we keep it general and do not impose a specific functional form on $\iota(m)$. As a consequence, the form of $M(\theta_t)$ is not specified either.

ideas is $e^{-p\theta_t}\nu_t$. Due to knowledge accumulation, whenever an idea is innovated new ones enter the stock. In period t there are a total of $(1 - e^{-p\theta_t})\nu_t$ innovations and their average quality is $M(\theta_t)$. So after knowledge accumulation takes place, the resulting stock of ideas is $e^{-p\theta_t}\nu_t + (1 - e^{-p\theta_t})M(\theta_t)\nu_t$. Lastly, varieties can become obsolete even before they are innovated. So at the end of period t a fraction δ of ideas suffer an obsolescence shock and exit the pool. Thus, the mass of research avenues entering period $t + 1$ is given by

$$\nu_{t+1} = (1 - \delta)\left[e^{-p\theta_t}\nu_t + \left(1 - e^{-p\theta_t}\right)M(\theta_t)\nu_t\right] \tag{8}$$

Higher tightness today maps into higher stock of ideas in the future. First, a high θ_t increases the likelihood any given idea is innovated. This raises the total number of innovations and, due to knowledge accumulation, the mass of research avenues tomorrow. Second, higher tightness increases the likelihood of simultaneous innovation. As a consequence, the average innovation quality, $M(\theta_t)$, is higher which leads to higher mass of ideas tomorrow.

Next, we focus on the law of motion for varieties. As shown in Figure 1, two events affect their stock: i) innovation; ii) an obsolescence shock. At the beginning of period t there are N_t varieties. During that period $(1 - e^{-p\theta_t})\nu_t$ ideas are invented. These ideas transform into new varieties and so after innovation takes place the resulting stock is $N_t + (1 - e^{-p\theta_t})\nu_t$. At the end of the period varieties become obsolete with probability δ. Thus, the stock entering period $t + 1$ is given by

$$N_{t+1} = (1 - \delta)\left[N_t + (1 - e^{-p\theta_t})\nu_t\right] \tag{9}$$

2.3 Households

There is a unit measure of infinitely lived identical consumers. They discount the future with a factor β and have the per-period utility function $U(C_t, L_t) = \ln C_t - \chi L_t^{1+1/\phi}/(1 + 1/\phi)$, where C_t is consumption, L_t is labor hours, ϕ is the Frisch elasticity of labor supply, and χ governs the disutility of labor. Since labor can be devoted to production or R&D, it follows that

$$L_t = L_{R_t} + L_{P_t} \tag{10}$$

Households own capital and have access to a mutual fund that covers all R&D firms. Let a_t denote the amount of shares held by the representative household at the beginning of period t. Firms distribute all profits as dividends, so the total assets of households in the beginning of t are $a_t \int_0^{N_t}(\pi_t(n) + V_t(n))dn + (1 + r_t)K_t$. At time t households choose shares a_{t+1} of the mutual fund which covers all R&D firms even though a fraction δ of varieties become obsolete next period. Thus, the household budget constraint is given by:

$$K_{t+1} + a_{t+1} \int_0^{\frac{N_{t+1}}{1-\delta}} V_t(n)dn = (1 + r_t)K_t + a_t \int_0^{N_t} (\pi_t(n) + V_t(n))dn + w_t L_t - C_t \tag{11}$$

2.4 Equilibrium

Intermediate good producers maximize per period profits subject to the inverse demand function given by equation (5). This yields $P_t = 1/\lambda$ and

$$X_t = \lambda\sigma \frac{Y_t}{N_t} \tag{12}$$

$$\pi_t = (1 - \lambda)\sigma \frac{Y_t}{N_t} \tag{13}$$

$$Y_t = (A_t(\sigma\lambda)^\sigma)^{\frac{1}{1-\sigma}} K_t^\alpha L_{P_t}^{1-\alpha} N_t^{\frac{\sigma(1-\lambda)}{\lambda(1-\sigma)}} \tag{14}$$

Since the production function is symmetric, holding a monopoly over any variety is equally profitable. Furthermore, profits depend on the amount of intermediate varieties, N_t, and on the concavity of the production function. If $\sigma/\lambda > 1$, then the production function exhibits increasing returns to scale and as a result profits are increasing in N_t. If, on the other hand, $\sigma/\lambda < 1$, then there are decreasing marginal returns to the extra variety and profits are decreasing in N_t.

At stage one of the innovation process, free entry implies that

$$\frac{\eta w_t}{A_t N_t} = \frac{1 - e^{-p\theta_t}}{\theta_t} V_t \tag{15}$$

Each entrant must hire $\eta/(A_t N_t)$ units of labor at the market wage w_t, so the left hand side of (15) captures the cost of engaging in R&D activities. Firms are successful in innovating with probability p. If they do innovate, they receive the patent with probability $(1-e^{-p\theta_t})/(p\theta_t)$. Thus, the right hand side of (15) captures the expected benefit from entering the R&D sector.

The first-order conditions of the representative household yield a standard labor supply condition and the Euler equations:

$$w_t = \chi C_t L_t^{\frac{1}{\phi}} \tag{16}$$

$$\frac{1}{C_t} = \beta E_t\left(\frac{1}{C_{t+1}}(1+r_{t+1})\right) \tag{17}$$

$$V_t = (1-\delta)\beta E_t\left(\frac{C_t}{C_{t+1}}(\pi_{t+1}+V_{t+1})\right) \tag{18}$$

where (18) makes use of the symmetry in varieties and their law of motion. Furthermore, recursive substitution yields the stochastic discount factor $d_{it} = \beta^{i-t}C_t/C_i$.

Lastly, we can combine the consumer's budget constraint, (11), with the demand for capital, (4), for labor in production, (3), and the free entry condition, (15), to get the law of motion for capital

$$K_{t+1} = (1-\delta^K)K_t + Y_t - X_tN_t - C_t \tag{19}$$

3 Numerical Results

3.1 Calibration

Following the previous literature (see, for example, Barlevy (2007)), calibrate the model at annual frequency. Hence, the discount factor is set at $\beta = 0.95$. The capital's share of output, α, is 0.33 and its depreciation rate, δ^K, is 0.08. The materials' share of output, σ, is set to 0.5 and the persistence parameter, ρ, to 0.88, as in Comin and Gertler (2006). Set the Frisch elasticity of labor supply, ϕ, to 4 and normalize $L_t = 1$. This yields $\chi = 0.8519$. Following Bilbiie *et al.* (2012), the obsolescence rate of varieties, δ, is set to 0.1. The probability of successfully innovating an idea, p, turns out to be a scaling parameter, so we normalize it to unity.

Since we study a loglinearized version of our model, we do not need to specify a functional form for $M(\theta_t)$. For our purposes it is sufficient to directly calibrate the value of $M(\theta_t)$ and its elasticity along the deterministic balanced growth path. We denoted these by $M(\theta)$ and $\varepsilon_{M,\theta}$ respectively. To pin down $M(\theta)$ along with the entry cost, η, and the gross markup, $1/\lambda$, we use three balanced growth path restrictions.[12] First, we use data on the fraction of approved patent applications in the U.S. for the period from 1966 to 2011.[13] We match its empirical average of 0.60957 to its model counterpart, $(1-e^{-p\theta})/(p\theta)$. This yields $\theta = 1.0876$. Second, the average R&D share of GDP for the U.S. is 3.1194%.[14] In our model aggregate R&D investment is $\eta w_t \mu_t/(N_t A_t)$ and the counterpart to GDP is final good production less the cost of producing intermediate goods, $Y_t - N_t X_t = (1-\lambda\sigma)Y_t$. Thus, we set $\eta w_t \mu_t/[(1-\lambda\sigma)Y_t N_t A_t] = 3.1194\%$. Third, we calibrate the growth rate of output to its empirical counterpart for the period of 1.7546%. This yields $\eta = 0.0563$, $\lambda = 0.9625$, and $M(\theta) = 1.7523$.

Lastly, to calibrate $\varepsilon_{M,\theta}$ and the volatility of the technology shock, σ_u, we use two second moment conditions. We set $\sigma_u = 0.01057$ to match the standard deviation of per capita non-farm GDP in the data of 2.7279%.[15] To match the standard deviation of per capita patent applications, $p\mu_t$, to its empirical counterpart of 3.9786%, calibrate $\varepsilon_{M,\theta} = 5.697$.

3.2 Model Solution

Our economy features aggregate uncertainty and positive long-run growth. As a result, it follows a stochastic BGP which fluctuates around the deterministic growth path. Here we provide a brief outline of how the model is solved. Further details are contained within the appendix. To obtain the numerical results, we follow Comin and Gertler (2006) and study a

[12]Further details on how the model is solved are provided in the next subsection and in the appendix.

[13]The data on both patents and patent applications is taken from the U.S. Patent and Trademark Office. The data on patent grants is by year of patent applications.

[14]The data is taken from the U.S. Bureau of Economic Analysis. The data on non-farm GDP is in chained 2009 dollars and is taken from NIPA table 1.3.6. The data for R&D expenditures is from NIPA table 5.6.5 and includes software expenditures. To deflate the series for R&D we use the implicit GDP price deflator from NIPA table 1.1.9.

[15]All per capita variables are normalized by the civilian non-institutionalized population. The data on this is taken from the Bureau of Labor Statistics' Employment Situation Release.

loglinearization of the stochastic growth path around its deterministic counterpart. To this end, first transform variables which feature positive growth into ratios that are constant along the deterministic BGP. Absent aggregate uncertainty, the growth rates of μ_t, ν_t, and N_t are $g_N = (1-\delta)(e^{-p\theta}+(1-e^{-p\theta})M(\theta))-1$ and those of Y_t, C_t, K_t are $(1+g_N)^{\sigma(1-\lambda)/(\lambda(1-\sigma)(1-\alpha))}-1$, where omitted time subscripts denote values along the deterministic BGP. The variables θ_t, L_t, L_{P_t}, and L_{R_t} exhibit no growth. Hence, we define $x_t := \nu_t/N_t$; $z_t := Y_t/C_t$; $n_t := K_t/C_t$; $\gamma_t := N_t^{\sigma(1-\lambda)/(\lambda(1-\sigma))}/K_t^{1-\alpha}$. Using the results in the previous section, we derive a system of nonlinear equations in the transformed variables, together with θ_t, A_t, L_t, L_{R_t} and L_{P_t}, that characterizes the economy's behavior along the stochastic BGP. Solving this system in the absence of aggregate uncertainty yields the solution for the deterministic BGP: $x = 0.7523$, $z = 2.4259$, $n = 2.6490$, $\gamma = 1.9640$, $\theta = 1.0876$, $A = L = 1$, $L_R = 0.0461$, $L_P = 0.9539$. Next, we loglinearize the system of equations around the deterministic BGP and substitute out $\tilde{L}_t$, $\tilde{z}_t$, $\tilde{L}_{R_t}$ and $\tilde{L}_{P_t}$, where tildes denote percentage deviations. This results in a system of five equations in $\tilde{x}_t$, $\tilde{\gamma}_t$, $\tilde{A}_t$, $\tilde{n}_t$, and $\tilde{\theta}_t$ given by $\mathbf{\tilde{y}_{t+1}} = \mathbf{J}\mathbf{\tilde{y}_t} + \tilde{\mathbf{e}}_{t+1}$, where $\mathbf{J}$ is a matrix of coefficients, $\mathbf{\tilde{y}_t} = [\tilde{x}_t\ \tilde{\gamma}_t\ \tilde{A}_t\ \tilde{\theta}_t\ \tilde{n}_t]'$, and $\tilde{\mathbf{e}}_{t+1}$ is a vector of error terms. The eigenvalues of $\mathbf{J}$ are $\mathrm{j}_1 = -0.9595$, $\mathrm{j}_2 = 0.8067$, $\mathrm{j}_3 = 0.88$, $\mathrm{j}_4 = 1.3404$, and $\mathrm{j}_5 = 3.8188$. The system has three initial conditions and three eigenvalues inside the unit circle. Hence, the rational expectations equilibrium is unique. Solving for that equilibrium yields $\tilde{\theta}_t = -0.3675\tilde{x}_t + 0.1075\tilde{\gamma}_t + 0.7760\tilde{A}_t$ and $\tilde{n}_t = -0.0010\tilde{x}_t - 0.7829\tilde{\gamma}_t - 1.3045\tilde{A}_t$. Lastly, use these equilibrium conditions to substitute out $\tilde{\theta}_t$ and $\tilde{n}_t$ from our system. Thus we are left with a 3×3 system of linear first-order difference equations in $\tilde{x}_t$, $\tilde{\gamma}_t$, and $\tilde{A}_t$. The numerical results are obtained from simulating that system.

The model is able to reproduce the empirically observed mild procyclicality of R&D because the economy features endogenous oscillations. Mathematically, these oscillations are driven by the negative sign of the eigenvalue j_1. To see this clearly, let Q be the matrix of eigenvectors, and Λ be the diagonal matrix of eigenvalues of $\mathbf{J}$. Then if $\tilde{\mathbf{x}}_t := Q^{-1}\tilde{\mathbf{y}}_t$, $\tilde{\mathbf{s}}_t := Q^{-1}\tilde{\mathbf{e}}_t$ our linearized system can be expressed as $\tilde{\mathbf{x}}_{t+1} = \Lambda\tilde{\mathbf{x}}_t + \tilde{\mathbf{s}}_{t+1}$. The first equation of this decoupled system is $\tilde{\mathbf{x}}^1_{t+1} = \mathrm{j}_1\tilde{\mathbf{x}}^1_t + \tilde{\mathbf{s}}^1_{t+1}$. Thus, a positive unit shock in the error term $\tilde{\mathbf{s}}^1_t$ leads to j_1^s units change in $\tilde{\mathbf{x}}^1_{t+s}$. Since $\mathrm{j}_1 < 0$, the sign of j_1^s alternates — when s is even the shock has a positive impact on $\tilde{\mathbf{x}}^1_{t+s}$ and when s is odd the impact is negative.

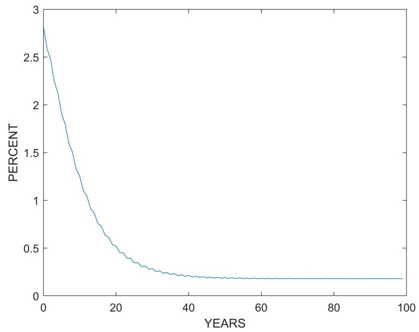
Hence, the series $\{\tilde{\mathbf{x}}^1_{t+s}\}_{s=0}^{\infty}$ will oscillate. When the magnitude of $\mathfrak{j}_1$ is large, shocks are more persistent and the oscillations more pronounced, when it is small the effect of the shock and the oscillations both dissipate quickly. The key parameter which governs both the sign and the magnitude of $\mathfrak{j}_1$ is the elasticity $\varepsilon_{M,\theta}$. When innovation quality is exogenous, i.e. $\varepsilon_{M,\theta} = 0$, the value of $\mathfrak{j}_1$ is 0.8035 and so the economy does not feature oscillations. As the elasticity increases, the eigenvalue decreases and eventually (for $\varepsilon_{M,\theta} \geq 1.63$) it turns negative. At that point oscillations emerge. As $\varepsilon_{M,\theta}$ increases further the oscillations increase in magnitude.

3.3 Impulse Response Functions

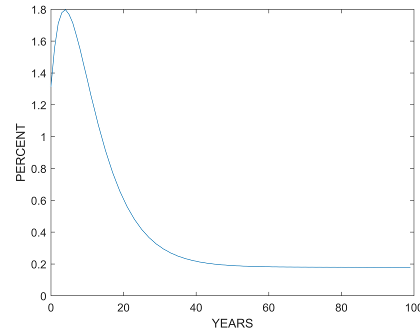
Because the model features endogenous growth, we construct the impulse response functions as percentage deviations from the deterministic balanced growth path of the economy (Figure 2). Following a positive technology shock, the economy converges to a new, higher BGP. Output, consumption, investment in R&D, labor hours, and profits initially increase and overshoot their new BGPs. In subsequent periods, they gradually converge to the higher BGP. Varieties, on the other hand, converge to the new BGP without initially overshooting it.

As is evident from the impulse response functions, the model exhibits endogenous oscillations. Whereas all variables feature this oscillating behavior, it is more pronounced in the ones that describe the R&D sector, i.e. the pool of ideas and the market tightness. Since the tightness, θ_t, is constant along the BGP, it converges to its old level. The main reason why the oscillations are more pronounced in the R&D sector is twofold. First, households want to smooth consumption and leisure, so they dislike volatility in total labor hours, output, spending on varieties, and total investment. Thus, the oscillation is least apparent in these variables. Second, the main driver behind the oscillations in the economy is the market for ideas, so variables associated with it exhibit higher magnitude oscillations.

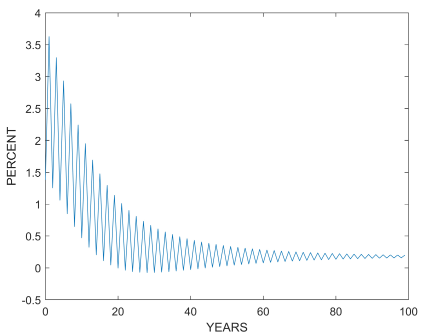
To see the intuition behind why the model exhibits endogenous oscillations, suppose that at period τ there are relatively more varieties, N_τ, and relatively fewer ideas, ν_τ. The scarcity of ideas implies that the number of innovations, $(1-e^{-p\theta_\tau})\nu_\tau$, is relatively low as well. This in turn leads to a low number of varieties the next period, $N_{\tau+1}$. Then, by equation



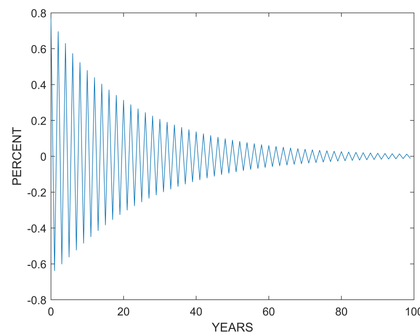
(a) Output



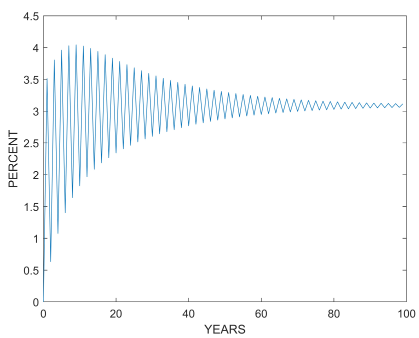
(b) Consumption



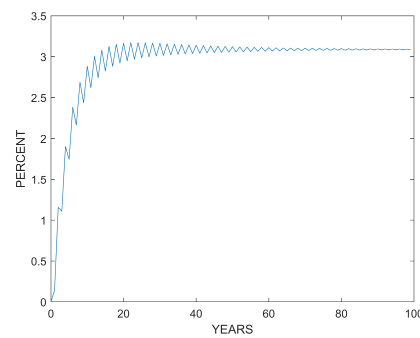
(c) R&D



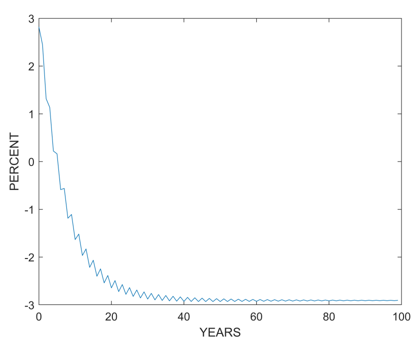
(d) Market Tightness



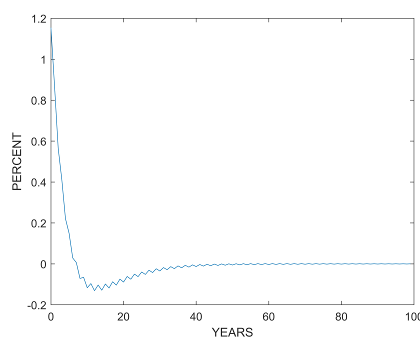
(e) Pool of Ideas



(f) Varieties



(g) Profits



(h) Hours

Figure 2: Impulse Response Functions to One Standard Deviation Shock in Productivity

(13), expected profits next period, $E_\tau \pi_{\tau+1}$, are high since there is relatively less competition between intermediate good producers.[16] At the same time, higher mass of varieties, N_τ, decreases the cost of innovation. Both the lower entry cost and higher expected profits create incentives for firms to enter the R&D sector. As a consequence the market tightness, θ_τ, increases. The resulting higher congestion implies that more firms, on average, innovate the same idea simultaneously. This leads to a higher average quality of innovation, $M(\theta_\tau)$, and as a consequence to relatively abundant research avenues the next period, $\nu_{\tau+1}$. Thus, at $\tau+1$ there are relatively low number of varieties, $N_{\tau+1}$, and relatively high number of ideas, $\nu_{\tau+1}$. Because of high $\nu_{\tau+1}$, the number of innovations at time $\tau+1$ and subsequently the number of varieties $N_{\tau+2}$ are relatively high. This leads to relatively low expected profits, $E_{\tau+1}\pi_{\tau+2}$. Because of lower expected profits and because of the relatively high entry cost at time $\tau+1$ (due to low $N_{\tau+1}$), firms have less of an incentive to innovate. This induces low market tightness, $\theta_{\tau+1}$. Hence, fewer firms innovate the same idea simultaneously which results in lower average quality of innovations. Thus, $\nu_{\tau+2}$ is relatively low. At time $\tau+2$ the cycle repeats.

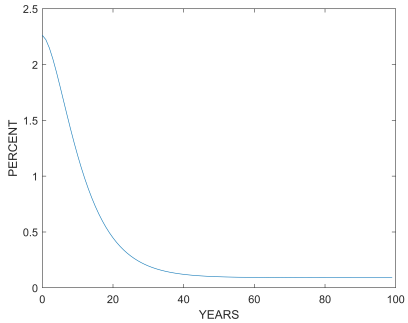
3.4 Cyclicality and Second Moments

Table 1: Moments for Data and Model

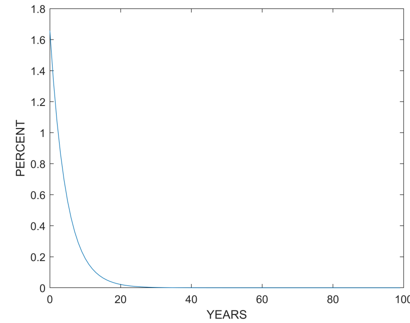
	σ_X/σ_Y			$\mathrm{Corr}(X,Y)$		
Variable X	Data	Benchmark Model	Exogenous Quality	Data	Benchmark Model	Exogenous Quality
R&D	1.79	1.89	0.81	0.43	0.44	0.99
Consumption	0.69	0.53	0.52	0.90	0.92	0.92
Hours	0.69	0.43	0.44	0.83	0.93	0.93
Investment	2.39	2.83	2.86	0.91	0.97	0.97

The endogenous oscillations in our model deliver mildly pro-cyclical R&D investment. This is in contrast to the existing literature which predicts near-perfect correlation between output and R&D.[17] Specifically, in periods when varieties are relatively abundant, output is relatively high because the final good firm can employ a wider range of intermediaries in

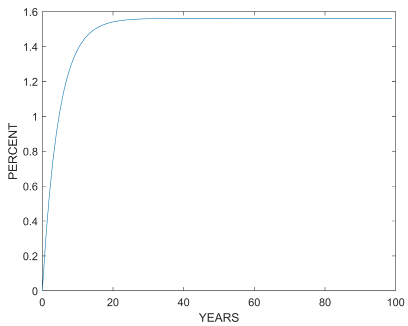
[16] Expected profits are decreasing in the number of varieties next period because in our calibration $\sigma < \lambda$.
[17] See, for example, Comin and Gertler (2006) and Francois and Lloyd-Ellis (2009).



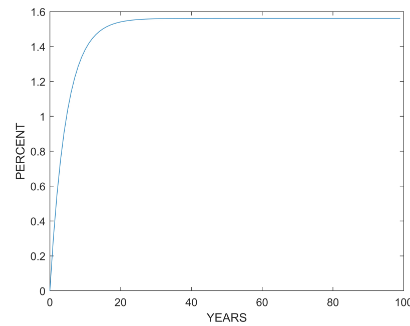
(a) R&D



(b) Market Tightness



(c) Pool of Ideas



(d) Varieties

Figure 3: Impulse Response Functions with Exogenous Innovation Quality

production. These periods also feature a low mass of available research avenues. As a result fewer firms engage in R&D.[18] This leads to low aggregate R&D investment. In contrast, during periods with relatively less varieties, output is low. At the same time, ideas are abundant, so entry into R&D, and consequently R&D investment, are high. Thus, although following a positive technology shock both output and R&D investment converge to a new, higher BGP, they oscillate around their convergent paths in such a way that whenever output overshoots its path, R&D undershoots it and vice versa.

Table 1 reports the standard deviations and cyclical properties of the data and our economy.[19] The model matches the empirical moments remarkably well. In particular, it is able to reproduce the mild procyclicality of R&D and its relative volatility. Alternative data sources for R&D yield a correlation with output between 0.3 and 0.5, and relative standard deviation of R&D around 1.9 times that of output.[20] The model does well against these

[18]This is true even though such periods feature a relatively high market tightness, θ_t.

[19]The data is obtained from the U.S. Bureau of Economic Analysis. Investment in the model corresponds to investment in physical capital and in R&D.

[20]Francois and Lloyd-Ellis (2009) use data from Compustat and find a correlation of 0.5 and relative

alternative measures as well.

The third and last columns of Table 1 highlight the importance of endogenous innovation quality for the model's ability to match the data. These columns report the moments for the model when innovation quality is exogenous, as captured by $M(\theta_t)$ being a constant, i.e. $\varepsilon_{M,\theta} = 0$. In that specification, the economy does not feature endogenous oscillations. Following a technology shock R&D investment, the tightness, the pool of ideas, and varieties all converge monotonically to their new BGP levels (see Figure 3). Hence, the mechanism which induces R&D to overshoot (undershoot) its convergent path whenever output undershoots (overshoots) it is broken. Thus, output and R&D move very closely together and their correlation is almost perfect. Furthermore, the absence of oscillations decreases the relative volatility of R&D by about one half.

4 Conclusion

This paper develops an expanding-variety endogenous growth model that can account for the empirically observed mild pro-cyclicality of R&D investment. In the model, some firms make the same innovation simultaneously. Innovations made by many firms simultaneously are, on average, of higher quality and so contribute more to the expansion of the knowledge stock in the economy. This mechanism gives rise to endogenous oscillations — periods of relatively scarce research avenues and abundant varieties are followed by periods during which research avenues are abundant and varieties scarce. Following a positive technology shock both output and R&D converge to a new, higher BGP. Thus, they are positively correlated. Due to the oscillations in the model, however, both variables fluctuate around their convergent paths in such a way that whenever R&D overshoots its path, output undershoots it and vice versa. Thus, R&D is only mildly pro-cyclical.

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5 Appendix

Here, we provide further details on the model solution method. First, we find the growth rates of model variables along the deterministic BGP.

Proposition 1. *Along the deterministic BGP θ_t, L_{P_t}, L_{R_t}, L_t are all constant. The growth rates of N_t, ν_t, and μ_t are $g_N = (1-\delta)(e^{-p\theta} + (1-e^{-p\theta})M(\theta)) - 1$, where θ is the value of θ_t along the deterministic BGP. The growth rates of Y_t, C_t, K_t are $(1+g_N)^{\sigma(1-\lambda)/(\lambda(1-\sigma)(1-\alpha))} - 1$.*

Proof. Let g_x denote the growth rate of any variable x along the deterministic BGP. First, it is clear that $g_{LP} = g_{LR} = g_L = 0$. Then, from (17) a constant growth rate of consumption implies that the return on capital, r_t, must be constant. Thus, from the final producer's first order condition with respect to capital, (4), we have that $g_Y = g_K$. Moreover, from the first order condition with respect to labor in production, (3), it follows that $g_w = g_Y$. Next, from the solution for intermediate goods, (12), and the law of motion for capital, (19), it follows that $g_K = -\delta^K + ((1-\lambda\sigma)Y_t - C_t)/K_t$. As g_K is constant along the BGP, it follows that $g_K = g_C$. As $g_K = g_Y$ and $g_{LP} = 0$, from the production function in equilibrium, (14), it follows that $g_Y = (1+g_N)^{\sigma(1-\lambda)/(\lambda(1-\sigma)(1-\alpha))} - 1$. Next, from the solution for profits, (13), it must be the case that $g_\pi = (1+g_Y)/(1+g_N) - 1$. Thus, from (18) and from the fact that g_C is constant, it follows that $g_\pi = g_V$. Then, free entry, (15), implies that $g_\theta = 0$. Hence, the growth rates of the mass of R&D entrants and ideas must be equal, $g_\mu = g_\nu$. Thus, the law of motion for varieties, (9), implies that $g_N = g_\nu$. Lastly, the law of motion for ideas, (8), implies that $g_N = (1-\delta)(e^{-p\theta} + (1-e^{-p\theta})M(\theta)) - 1$.

■

Given the results in Proposition 1, the ratios x_t, z_t, n_t, γ_t as defined in the text are stable. Using this transformation, the stochastic BGP is summarized by a system of nonlinear first order difference equations in x_t, n_t, γ_t, A_t, θ_t and equilibrium conditions for z_t, L_t, L_{P_t}, L_{R_t}.

The first difference equation in our system is simply the evolution of the technology parameter A_t, (2). We derive the rest as follows. Combine (8) with (9) and (9) with (19) to find the laws of motion for x_t and γ_t, respectively. From (4), (19), and (17), we derive (22) — a difference equation for n_t. The equation for θ_t, (23), we derive by combining (18), (15), (13), (9), and (3).

$$x_{t+1} = \frac{\left[e^{-p\theta_t} + (1 - e^{-p\theta_t})M(\theta_t)\right] x_t}{1 + (1 - e^{-p\theta_t})x_t} \tag{20}$$

$$\gamma_{t+1} = \frac{(1-\delta)^{\frac{\sigma(1-\lambda)}{\lambda(1-\sigma)}}(1 + (1 - e^{-p\theta_t})x_t)^{\frac{\sigma(1-\lambda)}{\lambda(1-\sigma)}} n_t^{1-\alpha}}{((1-\delta^K)n_t + (1-\lambda\sigma)z_t - 1)^{1-\alpha}}\gamma_t \tag{21}$$

$$\beta E_t\left[n_{t+1}\left(1 + \alpha(1-\sigma)\frac{z_{t+1}}{n_{t+1}} - \delta^K\right)\right] = (1-\delta^K)n_t + (1-\lambda\sigma)z_t - 1 \tag{22}$$

$$\beta E_t\left(z_{t+1}\Big(\frac{(1-\lambda)\sigma}{(1-\alpha)(1-\sigma)} + \frac{\eta\theta_{t+1}}{A_{t+1}L_{P_{t+1}}(1 - e^{-p\theta_{t+1}})}\Big)\right) = \frac{\eta\theta_t(1 + (1 - e^{-p\theta_t})x_t)z_t}{A_t L_{P_t}(1 - e^{-p\theta_t})} \tag{23}$$

The first equilibrium condition is the labor resource constraint, (10). Equations (24) and (25) we obtain by rearranging (6) and (14). The last condition, (26), follows from combining (3) and (16)

$$x_t\theta_t = A_t L_{R_t}\eta^{-1} \tag{24}$$

$$z_t = (A_t(\sigma\lambda)^\sigma)^{\frac{1}{1-\sigma}} n_t L_{P_t}^{1-\alpha}\gamma_t \tag{25}$$

$$\chi L_t^{\frac{1}{\phi}} L_{P_t} = (1-\alpha)(1-\sigma)z_t \tag{26}$$

Equations (20) through (26) along with (2) and (10) form the system which governs the economy's behavior along the stochastic BGP. Solving this system in the absence of aggregate uncertainty yields the solution for the deterministic BGP.

Next, we loglinearize the stochastic system around the deterministic growth path and substitute out $\tilde{z}_t$, $\tilde{L}_t$, $\tilde{L}_{R_t}$, $\tilde{L}_{P_t}$. This results in the liner first-order difference equation system

$$A^* \begin{bmatrix} \tilde{x}_t \\ \tilde{\gamma}_t \\ \tilde{A}_t \\ \tilde{\theta}_t \\ \tilde{n}_t \end{bmatrix} = B^* \begin{bmatrix} \tilde{x}_{t+1} \\ \tilde{\gamma}_{t+1} \\ \tilde{A}_{t+1} \\ E_t\tilde{\theta}_{t+1} \\ E_t\tilde{n}_{t+1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \tilde{u}_{t+1} \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

where

$$A^* = \begin{bmatrix} 0.6672 & 0 & 0 & 4.4159 & 0 \\ 0.0172 & 0.3186 & -1.3671 & 0.0114 & -0.6172 \\ 0 & 0 & 0.8800 & 0 & 0 \\ 0.1195 & 0.1477 & -0.0590 & 0.2246 & 0.1477 \\ -0.0171 & 2.7415 & 5.5001 & -0.0171 & 5.1786 \end{bmatrix}$$

and

$$B^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -0.0003 & 0.3745 & 0.5261 & 0.0995 & 0.3745 \\ -0.0052 & 0.8284 & 1.6620 & -0.0052 & 3.1437 \end{bmatrix}$$

Rearranging the system yields

$$\tilde{\mathbf{y}}_{t+1} = \mathbf{J}\tilde{\mathbf{y}}_t + \tilde{\mathbf{e}}_{t+1} \tag{28}$$

where $\tilde{\mathbf{y}}_t$ is as defined in the text, $\mathbf{J} := (B^*)^{-1}A^*$, and $\tilde{\mathbf{e}}_{t+1} := [b_{13}\tilde{u}_{t+1} \quad b_{23}\tilde{u}_{t+1} \quad b_{33}\tilde{u}_{t+1} \quad b_{43}\tilde{u}_{t+1} - \omega^{\theta}_{t+1} \quad b_{53}\tilde{u}_{t+1} - \omega^{n}_{t+1}]'$, where $b_{13} = 0$, $b_{23} = 0$, $b_{33} = 1$, $b_{34} = -3.2757$, $b_{53} = -0.5341$ are the elements of $(B^*)^{-1}$ and $\omega^{\theta}_{t+1} = E_t\tilde{\theta}_{t+1} - \tilde{\theta}_{t+1}$, $\omega^{n}_{t+1} = E_t\tilde{n}_{t+1} - \tilde{n}_{t+1}$.

This system has three initial conditions (x_0, γ_0, A_0 are given since N_0, ν_0, K_0, A_0 are predetermined). Given our calibration, the matrix $\mathbf{J}$ has exactly three eigenvalues inside the unit circle. Thus, the economy features a unique rational expectations equilibrium. Solving for this equilibrium yields the expressions for $\tilde{\theta}_t$ and $\tilde{n}_t$ shown in the text. Substituting for the rational expectations equilibrium conditions in (27) results in a 3×3 linear system in $\tilde{x}_t$, $\tilde{\gamma}_t$, $\tilde{A}_t$. Simulating that system yields the numerical results presented in the text.